The multi-birth property of Markov branching processes with immigration

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Background

- Branching process

State space $\mathbb{Z}_+ = \{0, 1, \cdots\}$.

▶ Definition

A conservative Q-matrix $Q = (q_{ij}; i, j \in \mathbb{Z}_+)$ is called a branching-immigration Q-matrix if it takes the following form:

$$q_{ij} = \begin{cases} ib_{j-i+1} + a_{j-i+1}, & \text{if } i \geq 0, j \geq i-1 \\ 0, & \text{otherwise,} \end{cases} \tag{1.1}$$

where

$$\begin{cases} a_0 = 0,\ a_j \geq 0\ (j \geq 2),\ 0 < -a_1 = \sum_{j=2}^{\infty} a_j < \infty, \\ b_j \geq 0\ (j \neq 1),\ 0 < -b_1 = \sum_{j \neq 1} b_j < \infty. \end{cases} . \tag{1.2}$$

Background

A Markov Branching-immigration process (simply, MBIP) is a continuous-time Markov chain taking values in $\mathbb{Z}_+$ whose transition function $P(t) = (p_{ij}(t) : i, j \in \mathbb{Z}_+)$ satisfies the Kolmogorov equations

$$P'(t) = P(t)Q, \tag{1.3}$$

where Q is a branching Q-matrix.

Background

Li and Chen [1] presented the regularity criteria for Q defined in (1.1)-(1.2). We assume that the process Q is regular throughout this talk.

Let $\{X(t) : t \geq 0\}$ denote the corresponding process and $P(t) = (p_{ij}(t) : i, j \in \mathbb{Z}_+)$ denote its transition function.

Define

$$F(t, u) = \sum_{j=0}^{\infty} p_{1j}(t)u^j.$$

Background

- Problems:

(i) How many particles died until time t ?

(ii) What is the m-birth number of particles until time t (here $m \neq 0$ is a fixed integer) ?

(iii) How many particles who ever lived in the system (i.e., the total death number)?

Background

- Related conclusions:

(i) Weighted branching process: Li Y., Li J. and Chen A. (2021, Sciences in China: Mathematics, in Chinese)

(ii) Weighted Markov collision processes: Li Y., Li J. (2021, Front. Math. China, 16(2):525 – 542).

Preliminary

We first make some preliminaries. Suppose that D is a finite subset of $\mathbb{Z}_+$ with $1 \notin D$. Let

$$[0,1]^D = \{\vec{v} = (v_k : k \in D) : v_k \in [0,1]\ \forall k \in D\}$$

and

$$\mathbb{Z}_+^D = \{\vec{l} = (l_k : k \in D) : l_k \in \mathbb{Z}_+\ \forall k \in D\}.$$

For simplicity of notations, in the following, we let $\mathbf{1}$ denote the vector in $\mathbb{Z}_+^D$ whose components are all 1 and for $k \in D$, $\vec{e}_k$ denote the vector in $\mathbb{Z}_+^D$ whose k'th component is 1 and others are 0.

Preliminary

Define

$$A(u) = \sum_{j=1}^{\infty} a_j u^{j-1}, \quad B(u) = \sum_{j=0}^{\infty} b_j u^j \tag{2.1}$$

and

$$B_D(u, \vec{v}) = \sum_{j \in D} b_j u^j \vec{v}^{\,\vec{e}_j}, \quad \bar{B}_D(u) = \sum_{j \in \bar{D}} b_j u^j \tag{2.2}$$

for $u \in [0,1], \vec{v} \in [0,1]^D$, where $\vec{v}^{\,\vec{l}} = \prod_{k \in D} v_k^{l_k}$ for $\vec{v} = (v_k : k \in D)$, $\vec{l} = (l_k : k \in D)$ and $\bar{D} = \mathbb{Z}_+ \setminus D$.

Preliminary

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for $u \in [0, 1], \vec{v} \in [0, 1]^D$, where $\vec{v}^{\,\vec{l}} = \prod_{k \in D} v_k^{l_k}$ for $\vec{v} = (v_k : k \in D)$, $\vec{l} = (l_k : k \in D)$ and $\bar{D} = \mathbb{Z}_+ \setminus D$.
It is obvious that $B(u)$, $\bar{B}_D(u)$ are well defined at least on $[0, 1]$, and $B_D(u, \vec{v})$ is well defined at least on $[0, 1] \times [0, 1]^D$.

Preliminary

The following theorem reveals the properties of $\bar{B}_D(u) + B_D(u, \vec{v})$.

Theorem 2.1.

(i) For any $\vec{v} \in [0,1]^D$,

$$\bar{B}_D(u) + B_D(u, \vec{v}) = 0 \tag{2.3}$$

has at most 2 roots in $[0,1]$. The minimal nonnegative root $\rho(\vec{v}) \leq \rho$, where ρ is the minimal nonnegative root of $B(u) = 0$.
(ii) $\lim_{\vec{v}\uparrow\mathbf{1}} \rho(\vec{v}) = \rho$, where $\vec{v} \uparrow \mathbf{1}$ means $v_k \uparrow 1$ $(k \in D)$.
(iii) $\rho(\vec{v}) \in C^\infty([0,1)^D)$ and $\rho(\vec{v})$ can be expanded as

$$\rho(\vec{v}) = \sum_{\vec{l} \in \mathbb{Z}_+^D} \rho_{\vec{l}}\, \vec{v}^{\,\vec{l}},$$

where $\rho_{\vec{l}} \geq 0$ for all $\vec{l} \in \mathbb{Z}_+^D$.

Preliminary

Sketch of proof.

(1) Since $0 \leq B_D(u, \mathbf{0}) \leq B_D(u, \vec{v}) \leq B_D(u, \mathbf{1})$, (i) follows from Li and Chen [2]. (ii) is easy.

(2) For (iii), it follows from Li, Li & Chen [3] that $\rho(\vec{v}) \in C^\infty([0,1)^D)$.

Preliminary

Suppose that

$$\rho(\vec{v}) = \sum_{\vec{k}\in\mathbb{Z}_+^N} \rho_{\vec{k}}\vec{v}^{\,\vec{k}}.$$

Substituting the above expression of $\rho(\vec{v})$ into (2.3) yields

$$0 \equiv \sum_{\vec{l}\in\mathbb{Z}_+^D} (\sum_{j\in\bar{D}} b_j\rho_{\vec{l}}^{*(j)})\vec{v}^{\,\vec{l}} + \sum_{j\in D} b_j \sum_{\vec{l}\in\mathbb{Z}_+^D} \rho_{\vec{l}}^{*(j)}\vec{v}^{\,\vec{l}+\vec{e}_j}.$$

(3) By using mathematical induction respect to $\vec{l}\cdot\mathbf{1}$, we can prove $\rho_{\vec{l}} \geq 0$. □

Conclusions

we now consider the multi-birth property of $\{X(t) : t \geq 0\}$.

As in the previous section, let $D \subset \mathbb{Z}_+$ be a finite subset with $1 \notin D$. We also assume that $b_k > 0$ for all $k \in D$ since there is no individual giving $(k-1)$ offsprings if $b_k = 0$. For simplicity of notation, we write the set $\{k-1 : k \in D\}$ as $D-1$ in the following, i.e.,

$$D - 1 = \{k - 1 : k \in D\}.$$

Conclusions

The main purpose of this talk is to analyze the $(D-1)$-birth numbers of $\{X(t): t \geq 0\}$. For this purpose, we construct a new Q-matrix $\widetilde{Q} = (\tilde{q}_{(i,\vec{m}),(j,\vec{l})} : (i,\vec{m}),\ (j,\vec{l}) \in \mathbb{Z}_+ \times \mathbb{Z}_+^D)$, where

$$\tilde{q}_{(i,\vec{m}),(j,\vec{l})} = \begin{cases} ib_{j-i+1} + a_{j-i+1}, & if\ \ i \geq 0, j-i+1 \in \bar{D}, \vec{l} = \vec{m}, \\ ib_{j-i+1}, & if\ \ i \geq 0, j-i+1 \in D, \vec{l} = \vec{m} + \vec{e}_{j-i+1}, \\ a_{j-i+1}, & if\ \ i \geq 0, j-i+1 \in D, \vec{l} = \vec{m}, \\ 0, & otherwise, \end{cases} \quad (3.1)$$

with $\{a_k : k \geq 0\}$ and $\{b_k : k \geq 0\}$ given in (1.2).

Conclusions

Let $\widetilde{P}(t) = (\tilde{p}_{(i,\vec{m}),(j,\vec{l})}(t) : (i,\vec{m}),(j,\vec{l}) \in \mathbb{Z}_+ \times \mathbb{Z}_+^D)$ be the Feller minimal $\widetilde{Q}$-function. Define

$$F_{i,\vec{m}}(t,u,\vec{v}) = \sum_{(j,\vec{l}) \in \mathbb{Z}_+ \times \mathbb{Z}_+^D} \tilde{p}_{(i,\vec{m}),(j,\vec{l})}(t) u^j \vec{v}^{\vec{l}}, \quad (u,\vec{v}) \in [0,1] \times [0,1]^D,$$

where $\vec{v}^{\vec{l}} = \prod_{k \in D} v_k^{l_k}$ for $\vec{v} = (v_k : k \in D)$ and $\vec{l} = (l_k : k \in D)$.

Lemma 3.1.

Let $\widetilde{Q}$ be defined in (3.1) and $\widetilde{P}(t) = (\tilde{p}_{(i,\vec{m}),(j,\vec{l})}(t))$ be the Feller minimal $\widetilde{Q}$-function. Then
(i) for any $(i, \vec{m}) \in \mathbb{Z}_+ \times \mathbb{Z}_+^D$ and $(u, \vec{v}) \in [0,1] \times [0,1]^D$,

$$\begin{aligned} \frac{\partial F_{i,\vec{m}}(t,u,\vec{v})}{\partial t} &= [\bar{B}_D(u) + B_D(u,\vec{v})] \cdot \frac{\partial F_{i,\vec{m}}(t,u,\vec{v})}{\partial u} \\ &\quad + A(u) \cdot F_{i,\vec{m}}(t,u,\vec{v}). \end{aligned} \tag{3.2}$$

Moreover,

$$\begin{aligned} F_{i,\vec{m}}(t,u,\vec{v}) - u^i \vec{v}^{\vec{m}} &= [\bar{B}_D(u) + B_D(u,\vec{v})] \cdot \frac{\partial}{\partial u} \mathbb{F}_{i,\vec{m}}(t,u,\vec{v}) \\ &\quad + A(u) \cdot \mathbb{F}_{i,\vec{m}}(t,u,\vec{v}), \end{aligned} \tag{3.3}$$

where $\bar{B}_D(u), B_D(u,\vec{v})$ are as in (2.2), $\mathbb{F}_{i,\vec{m}}(t,u,\vec{v}) = \int_0^t F_{i,\vec{m}}(s,u,\vec{v})ds$.
(ii) $\widetilde{Q}$ is regular if and only if Q is regular.

Conclusions

Sketch of proof. (1) By Kolmogorov forward equations,

$$\sum_{(j,\vec{l})\in\mathbb{Z}_+\times\mathbb{Z}_+^D} \tilde{p}'_{(i,\vec{m}),(j,\vec{l})}(t)u^j\vec{v}^{\vec{l}}$$

$$= [\bar{B}_D(u)+B_D(u,\vec{v})]\cdot \sum_{(k,\vec{r})\in\mathbb{Z}_+\times\mathbb{Z}_+^D} \tilde{p}_{(i,\vec{m}),(k,\vec{r})}(t)\cdot ku^{k-1}\vec{v}^{\vec{r}}$$

$$+A(u)\cdot \sum_{(k,\vec{r})\in\mathbb{Z}_+\times\mathbb{Z}_+^D} \tilde{p}_{(i,\vec{m}),(k,\vec{r})}(t)\cdot u^k\vec{v}^{\vec{r}}.$$

Thus, (i) is proved.

Conclusions

(2) Suppose Q is regular. By Li and Chen [1], we have $\rho = 1$ or that $\rho < 1$ and $\int_{\varepsilon}^{1} \frac{du}{-B(u)} = +\infty$ for all $\varepsilon \in (\rho, 1)$. If $\rho = 1$, then let $y = \rho(\vec{v})$ in (3.3), we know that

$$F_{i,\vec{m}}(t, \rho(\vec{v}), \vec{v}) - \rho^{i}(\vec{v})\vec{v}^{\vec{m}} = A(\rho(\vec{v})) \cdot \mathbb{F}_{i,\vec{m}}(t, \rho(\vec{v}), \vec{v}).$$

Then, letting $\vec{v} \uparrow \mathbf{1}$ in the above equality yields that $\widetilde{Q}$ is regular. If $\rho < 1$ and $\int_{\varepsilon}^{1} \frac{du}{-B(u)} = +\infty$ for all $\varepsilon \in (\rho, 1)$. Using Laplace transform, we can also get the conclusion.

Conclusions

(3) Conversely, suppose that $\widetilde{Q}$ is regular. By the theory of Markov chains $(\tilde{p}_{(i,\vec{m}),(j,\vec{l})}(t) : (i,\vec{m}),(j,\vec{l}) \in \mathbb{Z}_+ \times \mathbb{Z}_+^D)$ can be obtained as follows.

$$\tilde{p}^{(n)}_{(i,\vec{m}),(j,\vec{l})}(t) = \begin{cases} \delta_{(i,\vec{m}),(j,\vec{l})} e^{-\tilde{q}_{(i,\vec{m})} t}, \\ \tilde{p}^{(0)}_{(i,\vec{m}),(j,\vec{l})}(t) + \int_0^t e^{-\tilde{q}_{(i,\vec{m})} s} \sum_{(k,\vec{r}) \neq (i,\vec{m})} \tilde{q}_{(i,\vec{m}),(k,\vec{r})} \cdot \tilde{p}^{(n-1)}_{(k,\vec{r}),(j,\vec{l})}(t-s) ds, \end{cases}$$

$$\tilde{p}_{(i,\vec{m}),(j,\vec{l})}(t) = \lim_{n\to\infty} \tilde{p}^{(n)}_{(i,\vec{m}),(j,\vec{l})}(t), \quad (i,\vec{m}),(j,\vec{l}) \in \mathbb{Z}_+ \times \mathbb{Z}_+^D,$$

let

$$f^{(n)}_{i,j}(t,\vec{m}) = \sum_{\vec{l} \in \mathbb{Z}_+^D} \tilde{p}^{(n)}_{(i,\vec{m}),(j,\vec{l})}(t), \quad n \geq 0.$$

It can be proved that $f^{(n)}_{i,j}(t,\vec{m})$ is dependent of $\vec{m}$ and converges to the Feller minimal Q-function, which implies that Q is regular. □

Conclusions

Since we have assumed that Q is regular, by Lemma 3.1, we can see that $\widetilde{Q}$ determines a unique $\widetilde{Q}$-process $\{(\tilde{X}(t), \vec{Y}(t)) : t \geq 0\}$, where $\vec{Y}(t) = (Y_k(t) : k \in D)$ counts the $(D-1)$-birth number of $\{\tilde{X}(t) : t \geq 0\}$. It follows from the proof of Lemma 3.1 that $\{\tilde{X}(t) : t \geq 0\}$ is the MBIP with generator Q and thus has the same distribution as $\{X(t) : t \geq 0\}$. Therefore, we still use $\{X(t) : t \geq 0\}$ to denote $\{\tilde{X}(t) : t \geq 0\}$ in the following, i.e., $\{(X(t), \vec{Y}(t)) : t \geq 0\}$ is the $\widetilde{Q}$-process, where $\{X(t) : t \geq 0\}$ is the MBIP and $\vec{Y}(t) = (Y_k(t) : k \in D)$ counts the $(D-1)$-birth number of $\{X(t) : t \geq 0\}$.

Conclusions

In particular,
(i) if $D = \{0\}$ then $Y_0(t)$ counts the death number of $\{X(t) : t \geq 0\}$ until time t;
(ii) if $D = \{i\}$ $(i \geq 2)$, then $Y_i(t)$ counts the $(i-1)$-birth number of $\{X(t) : t \geq 0\}$ until time t;
(iii) if $D = \{0, i\}$ $(i \geq 2)$, then $\vec{Y}(t) = (Y_0(t), Y_i(t))$ counts the death number and the $(i-1)$-birth number of $\{X(t) : t \geq 0\}$ until time t.

Conclusions

Lemma 3.2.

For $\widetilde{P}(t)$, we have that for any $(i, \vec{m}) \in \mathbb{Z}_+ \times \mathbb{Z}_+^D$ and $(u, \vec{v}) \in [0,1] \times [0,1]^D$,

$$F_{i,\vec{m}}(t,u,\vec{v}) = [F_{1,\mathbf{0}}(t,u,\vec{v})]^i \cdot F_{0,\mathbf{0}}(t,u,\vec{v}) \cdot \vec{v}^{\vec{m}}, \tag{3.4}$$

where $\vec{v}^{\vec{m}} = \prod_{k \in D} v_k^{m_k}$ for $\vec{v} = (v_k : k \in D)$ and $\vec{m} = (m_k : k \in D)$.

Proof. Omitted.

Conclusions

Now, denote

$$\begin{cases} H(t,u,\vec{v}) = F_{0,\mathbf{0}}(t,u,\vec{v}), & (u,\vec{v}) \in [0,1] \times [0,1)^D, \\ G(t,u,\vec{v}) = F_{1,\mathbf{0}}(t,u,\vec{v}), & (u,\vec{v}) \in [0,1] \times [0,1)^D. \end{cases}$$

Lemma 3.3.

Suppose that $(u,\vec{v}) \in [0,1] \times [0,1)^D$. Then $(H(t,u,\vec{v}), G(t,u,\vec{v}))$ is the unique solution of the system of differential equations

$$\begin{cases} \frac{\partial x}{\partial t} = xA(y), \\ \frac{\partial y}{\partial t} = x[B_D(y,\vec{v}) + \bar{B}_D(y) + yA(y)], \\ x\mid_{t=0} = 1, \\ y\mid_{t=0} = u. \end{cases} \tag{3.5}$$

Conclusions

Sketch of proof. It can be proved by using Kolmogorov backward equations and Lemma 3.2.

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The following theorem gives the joint probability generating function of $(D-1)$-birth numbers until time t, i.e., the joint probability generating function of $\vec{Y}(t)$.

Conclusions

Theorem 3.1.

Suppose that $\{X(t) : t \geq 0\}$ is an MBIP. Then the joint probability generating function of $\vec{Y}(t)$ is given by

$$\begin{cases} E[\vec{v}^{\vec{Y}(t)}|X(0)=0] = H(t,1,\vec{v}), & \vec{v} \in [0,1]^D, \\ E[\vec{v}^{\vec{Y}(t)}|X(0)=1] = G(t,1,\vec{v}), & \vec{v} \in [0,1]^D, \end{cases}$$

where $(H(t,u,\vec{v}), G(t,u,\vec{v}))$ is the unique solution of (3.5).

In particular, if $a_1 = 0$, then

$$E[\vec{v}^{\vec{Y}(t)}|X(0)=1] = G(t,1,\vec{v}), \quad \vec{v} \in [0,1]^D,$$

where $G(t,u,\vec{v})$ is the unique solution of

$$\begin{cases} \frac{\partial y}{\partial t} = B_D(y,\vec{v}) + \bar{B}_D(y) \\ y|_{t=0} = u. \end{cases} \tag{3.6}$$

Conclusions

Furthermore,

$$P(\vec{Y}(t)=\vec{m}|X(0)=1)=g_{\vec{m}}(t)\ \forall \vec{m}\in \mathbb{Z}_+^D,$$

where

$$\begin{cases} g_{\mathbf{0}}(t)=G(t,1,\mathbf{0}) \\ g_{\vec{m}}(t)=\bar{B}_D(g_{\mathbf{0}}(t))\cdot\int_0^t \frac{F_{\vec{m}}(s)}{\bar{B}_D(g_{\mathbf{0}}(s))}ds,\ \ \vec{m}\neq\mathbf{0} \end{cases}$$

with

$$\begin{aligned} F_{\vec{m}}(t) &= \sum_{i\in D} b_i\cdot g^{*(i)}_{\vec{m}-\vec{e}_i}(t) \\ &\quad +\sum_{i\in\bar{D}} b_i\cdot \sum_{\vec{l}^{(1)},\cdots,\vec{l}^{(i)}\neq\vec{m},\ \vec{l}^{(1)}+\cdots+\vec{l}^{(i)}=\vec{m}} g_{\vec{l}^{(1)}}(t)\cdots g_{\vec{l}^{(i)}}(t) \end{aligned}$$

and $\{g^{*(i)}_{\vec{m}}(t):\vec{m}\in\mathbb{Z}_+^D\}$ being the i'th convolution of $\{g_{\vec{m}}(t):\vec{m}\in\mathbb{Z}_+^D\}$.

Conclusions

Sketch of Proof. (1) By Lemmas 3.2 and 3.3, we can prove (i).
(2) Suppose that $a_1 = 0$. (3.5) becomes (3.6). we suppose that

$$G(t, 1, \vec{v}) = \sum_{\vec{l} \in \mathbb{Z}_+^D} g_{\vec{l}}(t) \vec{v}^{\vec{l}}.$$

By (3.6), we get

$$\begin{cases} g'_{\mathbf{0}}(t) = \sum_{i \in \bar{D}} b_i g^i_{\mathbf{0}}(t) = \bar{B}_D(g_{\mathbf{0}}(t)), \\ g'_{\vec{l}}(t) = \sum_{i \in D} b_i g^{*(i)}_{\vec{l}-\vec{e}_i}(t) + \sum_{i \in \bar{D}} b_i g^{*(i)}_{\vec{l}}(t), \quad \vec{l} \neq \mathbf{0}. \end{cases} \tag{3.7}$$

Hence, it can be proved that

$$g_{\vec{l}}(t) = \bar{B}_D(g_{\mathbf{0}}(t)) \cdot \int_0^t \frac{F_{\vec{l}}(s)}{\bar{B}_D(g_{\mathbf{0}}(s))} ds, \quad \vec{l} \neq \mathbf{0}.$$

□

Conclusions

Remark 3.1.

(i) Generally, if $X(t)$ starts from $X(0) = i(> 1)$, then

$$E[\vec{v}^{\vec{Y}(t)} | X(0) = i] = H(t, 1, \vec{v}) \cdot [G(t, 1, \vec{v})]^i.$$

(ii) If $a_1 = 0$, then by the proof of Theorem 3.1,

$$G(t, u, \vec{v}) = \sum_{(j,\vec{l}) \in \mathbb{Z}_+ \times \mathbb{Z}_+^D} g_{j,\vec{l}}(t) u^j \vec{v}^{\vec{l}}, \quad (u, \vec{v}) \in [0, 1] \times [0, 1)^D,$$

where $g_{j,\vec{l}}(t) = p_{(1,\mathbf{0}),(j,\vec{l})}(t)$.

Conclusions

The following theorem gives a recursive algorithm of $g_{j,\vec{l}}(t)$.

Let $g_{j\vec{k}}^{*(i)}(t)$ be the ith convolution of $g_{j\vec{k}}(t)$ and

$$\begin{aligned} F_{j,\vec{k}}(t) &= \sum_{i\in\mathbb{N}} b_i g_{j\vec{k}-\vec{e}_i}^{*(i)}(t) \\ &+ \sum_{i\in\mathbb{N}^c} b_i \sum_{(l_1,\vec{k}_1),\cdots,(l_i,\vec{k}_i)\neq(j,\vec{k}),\sum_{m=1}^{i}(l_m,\vec{k}_m)=(j,\vec{k})} g_{l_1\vec{k}_1}(t)\cdots g_{l_i\vec{k}_i}(t). \end{aligned}$$

Conclusions

Theorem 3.2.

(i) If $0 \in D$ or $b_0 = 0$, then

$$\begin{cases} g_{0,\mathbf{0}}(t) = 0 \\ g_{j,\vec{l}}(t) = e^{b_1 t}[\delta_{j,1}\delta_{\vec{l},\mathbf{0}} + \int_0^t F_{j,\vec{l}}(s)e^{-b_1 s}ds], \quad (j,\vec{l}) \neq (0,\mathbf{0}). \end{cases}$$

(ii) If $0 \notin D$ and $b_0 > 0$, then

$$\begin{cases} g_{0,\mathbf{0}}(t) = G(t,0,\mathbf{0}) \\ g_{j,\vec{l}}(t) = \bar{B}_D(g_{0,\mathbf{0}}(t)) \cdot [\delta_{j,1}\delta_{\vec{l},\mathbf{0}} b_0^{-1} + \int_0^t \frac{F_{j,\vec{l}}(s)}{\bar{B}_D(g_{0,\mathbf{0}}(s))} ds], \quad (j,\vec{l}) \neq (0,\mathbf{0}), \end{cases}$$

Conclusions

Sketch of proof. Suppose that

$$G(t,u,\vec{v}) = \sum_{(j,\vec{k})\in\mathbb{Z}_+\times\mathbb{Z}_+^D} g_{j\vec{k}}(t)u^j\vec{v}^{\vec{k}}.$$

By (3.6),

$$\begin{aligned}\sum_{(j,\vec{k})\in\mathbb{Z}_+\times\mathbb{Z}_+^D} g'_{j\vec{k}}(t)u^j\vec{v}^{\vec{k}} &= \sum_{(j,\vec{k})\in\mathbb{Z}_+\times\mathbb{Z}_+^D\setminus\{\mathbf{0}\}}\sum_{i\in D} b_i g^{*(i)}_{j\vec{k}-\vec{e}_i}(t)u^j\vec{v}^{\vec{k}} \\ &+ \sum_{(j,\vec{k})\in\mathbb{Z}_+\times\mathbb{Z}_+^D}\sum_{i\in\bar{D}} b_i g^{*(i)}_{j\vec{k}}(t)u^j\vec{v}^{\vec{k}}.\end{aligned}$$

Conclusions

Comparing the coefficients on the both sides yields

$$g'_{j\vec{k}}(t) = \sum_{i\in D} b_i g^{*(i)}_{j\vec{k}-\vec{e}_i}(t) + \sum_{i\in \bar{D}} b_i g^{*(i)}_{j\vec{k}}(t), \quad (j,\vec{k}) \in \mathbb{Z}^{N+1}_{+}. \qquad (3.8)$$

Hence,

$$g_{0\mathbf{0}}(t) = G(t, 0, \mathbf{0}).$$

For $(j,\vec{k}) \neq (0,\mathbf{0})$, by (3.8),

$$g'_{j\vec{k}}(t) = g_{j\vec{k}}(t)\bar{B}'_D(g_{0\mathbf{0}}(t)) + F_{j,\vec{k}}(t). \qquad (3.9)$$

(i) If $0 \in D$ or $b_0 = 0$, then by (3.8), it is easy to see that

$$g_{0\mathbf{0}}(t) = 0, \quad \bar{B}'_D(g_{0\mathbf{0}}(t)) = b_1.$$

By (3.9),

$$g_{j\vec{k}}(t) = e^{b_1 t}[\delta_{j,1}\delta_{\vec{k},\mathbf{0}} + \int_0^t F_{j,\vec{k}}(s)e^{-b_1 s}ds].$$

(ii) If $0 \notin D$ and $b_0 > 0$, then

$$e^{\int_0^t \bar{B}'_D(g_{0\mathbf{0}}(s))ds} = e^{\int_0^t \bar{B}'_D(g_{0\mathbf{0}}(s))\cdot \frac{g'_{0\mathbf{0}}(s)}{\bar{B}_D(g_{0\mathbf{0}}(s))}ds} = \frac{\bar{B}_D(g_{0\mathbf{0}}(t))}{b_0}$$

Hence,

$$g_{j\vec{k}}(t) = \bar{B}_D(g_{0\mathbf{0}}(t)) \cdot [\delta_{j,1}\delta_{\vec{k},\mathbf{0}}b_0^{-1} + \int_0^t \frac{F_{j,\vec{k}}(s)}{\bar{B}_D(g_{0\mathbf{0}}(s))}ds], \quad (j,\vec{k}) \neq (0,\mathbf{0}).$$

Conclusions

Corollary 3.1.

Let $\{X(t); t \geq 0\}$ be an MBP with $X(0) = 1$. Then

$$E[v^{Y_0(t)}|X(0) = 1] = G(t, 1, v), \quad v \in [0, 1],$$

where $G(t, u, v)$ is the unique solution of the equation

$$\begin{cases} \frac{\partial y}{\partial t} = B(y) - b_0(1 - v), \\ y|_{t=0} = u, \end{cases} \quad u, v \in [0, 1].$$

Conclusions

Corollary 3.2.

Let $\{X(t); t \geq 0\}$ be an MBP with $X(0) = 1$ and $m(> 1)$ be fixed. Then

$$E[v^{Y_m(t)}|X(0) = 1] = G(t, 1, v), \quad v \in [0, 1],$$

where $G(t, u, v)$ is the unique solution of the equation

$$\begin{cases} \frac{\partial y}{\partial t} = B(y) - b_m(1 - v)y^m, \\ y|_{t=0} = u, \end{cases} \quad u, v \in [0, 1].$$

Conclusions

Let

$$\tau = \inf\{t \geq 0 : X(t) = 0\}$$

be the hitting time of 0 for $X(t)$.
By Theorem 3.1, we have

Theorem 3.3.

Let $\{X(t) : t \geq 0\}$ be an MBP with $X(0) = 1$. Then

$$E[\vec{v}^{\vec{Y}(\tau)} | \tau < \infty] = \rho^{-1} \cdot \rho(\vec{v}), \quad \vec{v} \in [0,1]^D,$$

where ρ is the minimal nonnegative root of $B(u) = 0$.

Conclusions

Sketch of proof. (1) By Theorem 3.1 and (3.3) with $i = 1$ and $u = \rho(\vec{v})$, for $\forall t \geq 0$,

$$\rho(\vec{v}) = \sum_{\vec{l} \in \mathbb{Z}_+^D} p_{(1,\mathbf{0}),(0,\vec{l})}(t)\vec{v}^{\vec{l}} + \sum_{\vec{l} \in \mathbb{Z}_+^D} (\sum_{j=1}^{\infty} p_{(1,\mathbf{0}),(j,\vec{l})}(t)\rho(\vec{v})^j)\vec{v}^{\vec{l}}. \quad (3.10)$$

(2) Further prove that

$$G(\infty, 1, \vec{v}) = \sum_{\vec{l} \in \mathbb{Z}_+^D} p_{(1,\mathbf{0}),(0,\vec{l})}(\infty)\vec{v}^{\vec{l}} + \lim_{t \to \infty} \sum_{\vec{l} \in \mathbb{Z}_+^D} (\sum_{j=1}^{\infty} p_{(1,\mathbf{0}),(j,\vec{l})}(t))\vec{v}^{\vec{l}} \quad (3.11)$$

and

$$\rho(\vec{v}) = \sum_{\vec{l} \in \mathbb{Z}_+^D} p_{(1,\mathbf{0}),(0,\vec{l})}(\infty)\vec{v}^{\vec{l}}. \quad (3.12)$$

Conclusions

(3) By (3.11) and (3.12),

$$G(\vec{v}) = \sum_{\vec{l}\in\mathbb{Z}_+^D} P(\vec{Y}(\tau) = \vec{l}\,|\tau < \infty) \cdot \vec{v}^{\vec{l}} = \rho^{-1} \cdot \rho(\vec{v})$$

and

$$P(\vec{Y}(\tau) \le \vec{l}\,|\tau = \infty) = (1-\rho)^{-1} \cdot \lim_{t\to\infty} \sum_{\vec{m}\le\vec{l}} \sum_{j=1}^{\infty} p_{(1,\mathbf{0}),(j,\vec{m})}(t) = 0.$$

$\square$

Examples

Example 3.1. Let $X(t)$ be a birth-death type MBP with death rate pb and birth rate qb (here, $b > 0,\ p \in (0,1), p+q=1$), $X(0) = 1$. Then

$$B(u) = b(p - u + qu^2).$$

Proposition 3.1.

Let $Y(t)$ be the death number of $X(\cdot)$ until t. Then

$$E[v^{Y(t)}] = \beta(v) + \frac{\alpha(v) - \beta(v)}{1 + \frac{\alpha(v)-1}{1-\beta(v)} \cdot e^{[\alpha(v)-\beta(v)]bqt}},$$

where

$$\alpha(v) = \frac{1 + \sqrt{1 - 4pqv}}{2q}, \qquad \beta(v) = \frac{1 - \sqrt{1 - 4pqv}}{2q}.$$

Examples

Proposition 3.2.

Let $Y(t)$ be the death number of $X(\cdot)$ until t. Then

$$E[v^{Y(\tau)}|\tau < \infty] = \beta(v),$$

where

$$\beta(v) \quad = \quad p\left(v + \sum_{n=2}^{\infty} \frac{(2n-3)!!2^{n-1}(pq)^{n-1}}{n!} v^n\right).$$

Examples

Example 3.2. Let $X(t)$ be an MBP with $b_0 = pb$ and $b_3 = qb$ (here, $b > 0,\ p \in (0,1), p+q=1$), $X(0) = 1.$ Then

$$B(u) = b(p - u + qu^3).$$

Examples

Proposition 3.3.

Let $Y(t)$ be the death number of $X(\cdot)$ until t. Then

$$E[v^{Y(t)}] = \sum_{n=0}^{\infty} g_n(t) v^n,$$

where

$$\begin{cases} g_0(t) = (q + pe^{2bt})^{-1/2} \\ g_n(t) = e^{2bt} \cdot (q + pe^{2bt})^{-3/2} \cdot \int_0^t e^{-2bs}(q + pe^{2bs})^{3/2} F_n(s) ds, \ \ n \geq 1 \end{cases}$$

with

$$F_n(t) = bp\delta_{1,n} + bq \cdot \sum_{k_1, k_2, k_3 < n, k_1 + k_2 + k_3 = n} g_{k_1}(t) g_{k_2}(t) g_{k_3}(t).$$

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